Dynamics of globally coupled bistable elements

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The macroscopic dynamics of a large set of globally coupled, identical, noiseless, bistable elements is analytically and numerically studied. Depending on the value of the coupling constant and on the initial condition, all the elements can either evolve towards the same individual state or become divided into two groups, which approach two different states. It is shown that at a critical value of the coupling constant the system undergoes a transition from bistable evolution, where the two behaviors described above can occur, to coherent evolution, where the convergence towards the same individual state is the only possible behavior. Connections of this system with the real Ginzburg-Landau equation and with the sociological problem of opinion formation are discussed. [S1063-651X(97)10505-0]

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I. INTRODUCTION

The study of complex behavior in extended systems is based on the analysis of large sets of elements whose local dynamics is coupled through some interaction mechanism. Much effort has been devoted in the last two decades to the investigation of extended systems with short-range interactions, whose main manifestations of complex behavior are by now well understood. Reaction-diffusion systems constitute the paradigm of this problem. More recently, other coupling mechanisms—in particular, global coupling [1]—have also been considered. Global coupling plays a relevant role in models of many real systems driven by long-range interactions, able to generate strong correlations between highly interconnected elements. Instances of such systems are oscillating catalytic surface reactions [2], neural networks [3,4], and allosteric enzymic reactions [5].

Forms of collective behavior produced by global coupling have been well characterized in the case of systems formed by limit-cycle oscillators. In these systems, long-range interactions can give rise to synchronized oscillations [1,6]. This kind of ordered entrained evolution—which has been observed in systems formed by either identical or slightly different elements—is part of a wide class of possible behaviors with nontrivial features, including clustering, chaotic collective dynamics, and desynchronization [7,8].

Although much attention has recently been paid to these sets of globally coupled oscillators, a full understanding of the role of global coupling in the dynamics of extended complex systems—to the levels already reached in the case of diffusive coupling—will require one to study other types of local dynamics. In particular, one should be interested in characterizing the forms of collective evolution that occur in systems of elements whose individual dynamics differ from limit-cycle oscillations. In this spirit, this paper is devoted to the analysis of the evolution of a set of globally coupled bistable elements. This should be relevant to the study of spin systems [9] and neural networks [10], where closely related models have already been considered.

Models of coupled bistable elements have been addressed in the study of critical phenomena under the effects of external noise [11]. Internal deterministic noise has also been considered in systems of coupled chaotic elements with similar symmetry properties [12]. In this frame, it has been shown that a ferromagneticlike transition occurs as the noise strength is varied. The same models, added with suitable harmonic forcing, have been very recently studied in connection with stochastic resonance in extended systems [13]. In this paper, instead, the attention is focused on qualitative changes in the macroscopic behavior of a set of noiseless bistable elements upon variation of the coupling strength. In Sec. II, the mathematical model is presented and it is suggested that a critical phenomenon takes place as a coupling constant is varied. Section III is devoted to the characterization of this critical phenomenon, which is a kind of first order phase transition. Finally, results are summarized and discussed in Sec. IV.

II. GLOBAL COUPLING OF BISTABLE ELEMENTS

Consider a set of N identical elements, each of them characterized by a state variable $x_i(t)$, with $-1 \leqslant x_i \leqslant 1$. In the absence of coupling the individual dynamics is bistable, and the evolution of x_i is governed by the equation

$$\dot{x} = x - x^3, \tag{1}$$

which corresponds to overdamped motion in the one-dimensional potential $V(x) = -x^2/2 + x^4/4$. The solution to Eq. (1) is

$$x(t) = \mathrm{sgn}(x_0)[1 - (1 - x_0)^{-2} \exp(-2t)]^{-1/2}, \tag{2}$$

with $x_0 = x(0)$. During the evolution, $x(t)$ preserves its sign, and approaches the asymptotic value $x|_{t\to\infty} = \mathrm{sgn}(x_0) = \pm 1$. The stationary state $x = 0$ is unstable.

Global coupling is now introduced in the usual way [1], as a term describing relaxation towards the mean value $\overline{x(t)} = N^{-1}\Sigma_i x_i(t)$:

$$\dot{x}_i=x_i-x_i^3+k(\overline{x}-x_i)=(1-k)x_i+k\overline{x}-x_i^3. \quad (3)$$

In the following, the (positive) coupling constant k is restricted to the interval [0,1]. In fact, as shown below, the behavior of the coupled system for $k=1$ is essentially the same as for larger values of the coupling constant. For large k, the evolution proceeds in two well-defined stages. Up to $t\approx k^{-1}$, the effect of coupling is dominant and each x_i rapidly approaches the mean value $\overline{x(t)}$. From then on, the coupled set evolves coherently—i.e., the states $x_i(t)$ of all the elements coincide, and $\overline{x(t)}=x_i(t)$ for all i. The subsequent behavior is thus mainly governed by the individual dynamics, and the whole set approaches one of the two stable states. Note that, as in the evolution of a single element, this asymptotic state is selected by the initial condition.

In the case of $k=1$, Eq. (3) reduces to

$$\dot{x}_i=\overline{x}-x_i^3. \quad (4)$$

Let $r=x_i-x_j$ be the difference between the states of any two elements in the system and, without loss of generality, suppose $r>0$. According to Eq. (4), this quantity satisfies

$$\dot{r}=-r(x_i^2+x_ix_j+x_j^2). \quad (5)$$

Now, since $-1\leqslant x_i,x_j\leqslant 1$, the inequalities $r^2/4\leqslant x_i^2+x_ix_j+x_j^2\leqslant 3$ hold and, at each time,

$$-3r\leqslant\dot{r}\leqslant-\frac{r^3}{4}. \quad (6)$$

The inequality signs are inverted if $r<0$. In the $(r,\dot{r})$ plane, therefore, the trajectory corresponding to the solution of Eq. (5) must lie between the graphs of the functions appearing in Eq. (6), which intersect each other at $r=0$. As a consequence, r vanishes for $t\to\infty$. Hence, for $k=1$ the coupled elements evolve coherently and a single asymptotic value of x_i is approached for all i.

In agreement with these results, numerical calculations show that, for $k\geqslant 1$, the system converges to coherent behavior as time elapses. On the other hand, for $k=0$ each element evolves independently according to Eq. (2) and, from a generic initial condition, the set becomes divided into two groups, each of them approaching one of the two stable states. Figure 1 shows the evolution of a set of 10^3 bistable elements in the cases $k=0$ and $k=1$—although, for the sake of clarity, only 10^2 trajectories are plotted. Both plots correspond to exactly the same initial uniform random distribution in $(-1,1)$.

The qualitative change in the evolution between $k=0$ and $k=1$ suggests that some kind of transition between both behaviors should occur at some intermediate value of the coupling constant. This transition is characterized in the following.

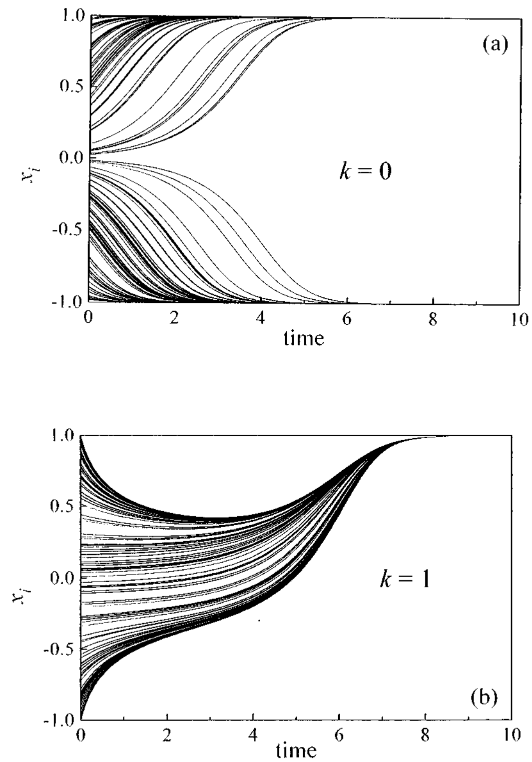

FIG. 1. Temporal evolution of the coordinates $x_i(t)$ for a set of bistable elements with random homogeneous initial distribution, $-1<x_i(0)<1$, and two values of the coupling constant, (a) $k=0$ and (b) $k=1$. For clarity, only 10^2 curves are displayed in each plot.

III. TRANSITION BETWEEN BISTABLE AND COHERENT BEHAVIOR

According to numerical calculations, for sufficiently small values of the coupling constant the global behavior of the system qualitatively reproduces the evolution of the set of uncoupled elements $(k=0)$. In fact, depending on the initial distribution of the coordinate x_i, all the elements converge to one of the extreme values $x_i=\pm 1$—as when, for $k=0$, all the coordinates have the same sign—or become divided into two groups, which approach two different values of the coordinate—as when, for $k=0$, both signs are present in the initial distribution. The coupled system is therefore ''bistable'' in the sense that two qualitatively different asymptotic states can be observed, depending on the inital condition: either the elements behave coherently, all of them approaching the same final state, or they are divided into two groups. On the other hand, as stated above, for larger values of k only coherent behavior is observed.

The transition between bistable and coherent behavior is characterized by a stability change in the possible asymptotic states of the whole system. Suppose that, as the system evolves, the N elements are divided into two groups. One of them, containing pN elements $(0<p<1)$ approaches the coordinate X_1, whereas the other, with $(1-p)N$ elements, approaches X_2. It has to be stressed that the value of p is determined—in a nontrivial way—by the initial condition. According to Eq. (3), the following identities should hold as $N\to\infty$:

$$\begin{aligned}0&=(1-k)X_1+k[pX_1+(1-p)X_2]-X_1^3,\\0&=(1-k)X_2+k[pX_1+(1-p)X_2]-X_2^3.\end{aligned} \quad (7)$$

Note that the case of coherent evolution can be taken into

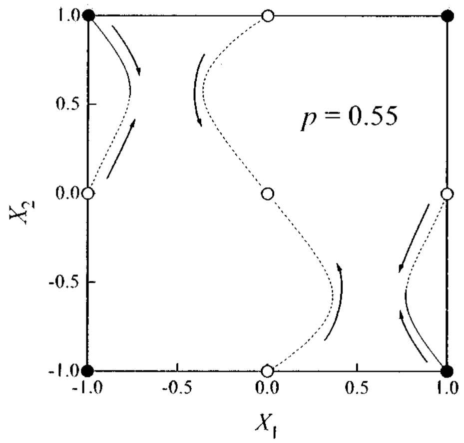

FIG. 2. Solutions to Eqs. (7) for $p=0.55$ and $0<k<1$. Full dots and lines correspond to stable solutions whereas empty dots and dashed lines stand for unstable states. The arrows indicate the direction of growing k.

account by putting $X_1=X_2$. The solutions to Eqs. (7) are, therefore, the whole set of stationary states of the coupled system. Their stability can be studied from Eq. (3) in the standard linear approximation.

Independently of the value of p, Eqs. (7) have nine solutions. The trivial one, $(X_1,X_2)=(0,0)$, is unstable. The remaining eight solutions can be grouped into symmetrical pairs, (X_1,X_2) and $(-X_1,-X_2)$, both with the same stability properties. It is therefore enough to analyze, for instance, the four solutions with $X_1 \geqslant 0$. (i) The first one, $(X_1,X_2)=(1,1)$, is stable and corresponds to the asymptotic state of coherent evolution. (ii) The second solution is real for all k. It approaches the unstable solution $(0,-1)$ for $k \to 0$ and the trivial solution (0,0) for $k=1$. This solution is unstable for all k. (iii) Another solution, which is also unstable for all k, approaches the unstable solution (1,0) as $k \to 0$. (iv) Finally, there is a stable solution that approaches $(1,-1)$ as $k \to 0$. This solution corresponds to the state in which the elements have become divided into two subsets.

Figure 2 shows the numerical calculation of the pairs (X_1,X_2) given by Eq. (7) for $p=0.55$ as the coupling constant varies for $k=0$ to $k=1$. Arrows indicate the direction of increasing k. Solid (dashed) lines and solid (open) circles stand for stable (unstable) equilibrium states.

As the coupling constant grows, there is a critical value k_c at which the two solutions (iii) and (iv) ''collide'' and become complex. At this critical value, then, the solution in which the whole set becomes divided into two groups dissapears. The value of k_c is related to p according to

$$1=4k_c-18k_c^2p+18k_c^2p^2+27k_c^4p^2-54k_c^4p^3+27k_c^4p^4. \quad (8)$$

Thus, for a given value of p—which is determined by the initial condition—and $k<k_c$ two qualitatively different behaviors can occur, as suggested by the numerical simulations. Either $X_1=X_2=\pm 1$ and the system evolves coherently, or $X_1 \neq X_2$ and the elements are divided into two groups. For $k>k_c$, instead, only the coherent evolution is possible. Figure 3 shows a phase diagram k versus p, where

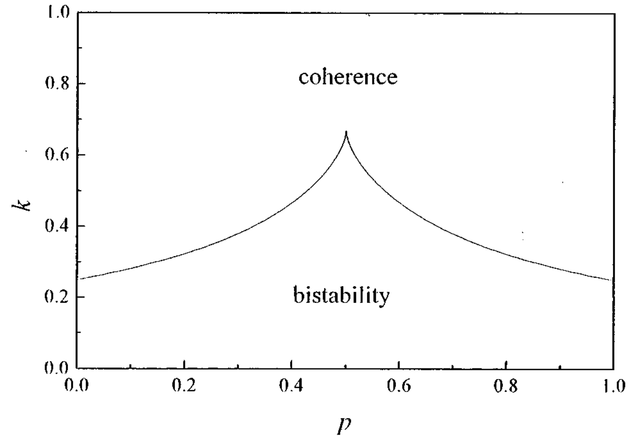

FIG. 3. Phase diagram in the (p,k) plane. The curve divides the zones of bistability—where according to the initial condition, the set of elements is divided into two groups or approaches a single value of x_i—and of coherent behavior, where all the elements are always attracted to the same state.

the boundary between the regions of bistability and coherence given by Eq. (8) has been plotted.

A characterization of this transition in terms of a single order parameter is achieved by introducing the mean square displacement of the asymptotic distribution of coordinates x_i with respect to their mean value $\bar{X}=pX_1+(1-p)X_2$, namely,

$$\sigma=\sqrt{p(X_1-\bar{X})^2+(1-p)(X_2-\bar{X})^2}=\sqrt{p(1-p)}|X_1-X_2|. \quad (9)$$

Figure 4 shows the value of σ as a function of k, for fixed p. Solid (dashed) lines stand for stable (unstable) states; the horizontal axis $\sigma=0$ corresponds to the stable states $X_1=X_2=\pm 1$. The dependence of σ on k suggests classifying the transition between bistable to coherent behavior as a subcritical first order transition. Note, however, that, since

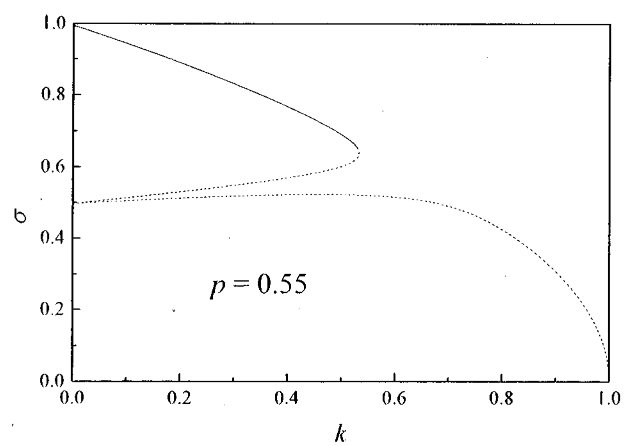

FIG. 4. Mean square dispersion for the solutions to Eq. (7) with $p=0.55$, as a function of the coupling constant k. Full (dotted) lines correspond to stable (unstable) states. Coherent evolution corresponds to $\sigma=0$.

coherent behavior is stable for any value of the coupling constant, hysteresis effects cannot occur.

As already stated, the ratios p and $1-p$ into which the set of coupled elements is divided into the bistable regime depend in a nontrivial way—which, unfortunately, cannot be fully described in analytical terms—on the initial condition. In order to illustrate this dependence, consider a set of elements initially distributed at random, according to a probability distribution given by

$$P(x_i)=\begin{cases}1-p_0 & \text{for } -1<x_i<0\\ p_0 & \text{for } 0<x_i<1,\end{cases} \tag{10}$$

with $0<p_0<1$. In this initial condition, a fraction p_0 $(1-p_0)$ of the elements have positive (negative) coordinate. In the absence of coupling, then, the asymptotic distribution corresponds to $p=p_0$.

Note that for $p_0=1/2$ the symmetry of the whole problem implies that, in the limit $N\to\infty$, the distribution of elements is symmetric along the entire evolution. In particular, $\overline{x(t)}=0$ for all t. As stated before, however, this state is unstable for $k>k_c|_{p=1/2}\approx 0.67$. Therefore, for any finite value of N and for $k>k_c$, the (statistical) symmetry of the initial condition will break down as time elapses and the mean value $\overline{x}$ will asymptotically approach a nonvanishing value, $\overline{x}=\pm 1$. Global coupling is thus able to amplify the microscopic fluctuations in the homogenous (but random) initial distribution, giving rise to an asymmetry at the macroscopic level.

In Fig. 5 the evolution of a set of 10^3 coupled elements is displayed for $p_0=0.55$ and two values of k. Both plots show 10^2 trajectories only, evolving from the same initial condition. For $k=0.45$, the asymptotic value $p\approx 0.58$ is reached. For $p=0.58$ the critical value of the coupling constant is $k_c\approx 0.48$. In fact, Fig. 5(b) shows that, for $k=0.55$, the system behaves coherently.

Finally, Fig. 6 shows the results of a series of numerical calculations in which a set of 10^3 coupled elements was built according to the initial condition (10). The asymptotic value of the ratio p was determined as a function of the coupling constant, averaging over some 10^2 realizations of the initial condition for each value of p_0 and k. As indicated before, the value of p_0 coincides with the value of p obtained for $k=0$. The sharp transition to coherent behavior ($p=1$) at a critical value of k is apparent.

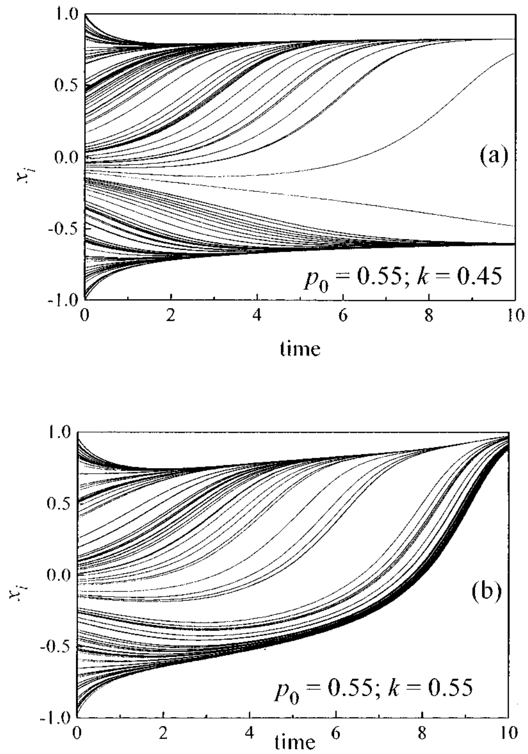

FIG. 5. Temporal evolution of the coordinates $x_i(t)$ for a set of bistable elements with random initial distribution as given in Eq. (10) for $p_0=0.55$ and two values of the coupling constant, (a) $k=0.45$ and (b) $k=0.55$. For clarity, only 10^2 curves are displayed in each plot.

IV. SUMMARY AND DISCUSSION

In this paper, the noiseless dynamics of a set of globally coupled identical bistable elements has been considered. The coupling mechanism is a kind of mean-field diffusionlike process, already considered in the literature in connection with other types of individual dynamics, especially oscillators and excitable elements [1,2,4,6–8]. It has been shown that the elements can either become divided into two groups, which approach two different states, or behave coherently and evolve towards a completely homogeneous state. Below a certain critical value for the coupling constant, and depending on the initial condition, both behaviors can occur and the system is therefore ''bistable.'' Above this value, instead, when the coupling is strong enough, only coherent behavior is possible. The transition between bistable and coherent evolution is similar to a first-order phase transition.

Coherent evolution of the bistable elements is qualitatively similar to the synchronization observed in globally coupled limit-cycle oscillators. In both cases, a sufficiently strong coupling forces each element to become entrained in the average motion of the set. Dissipative effects then make

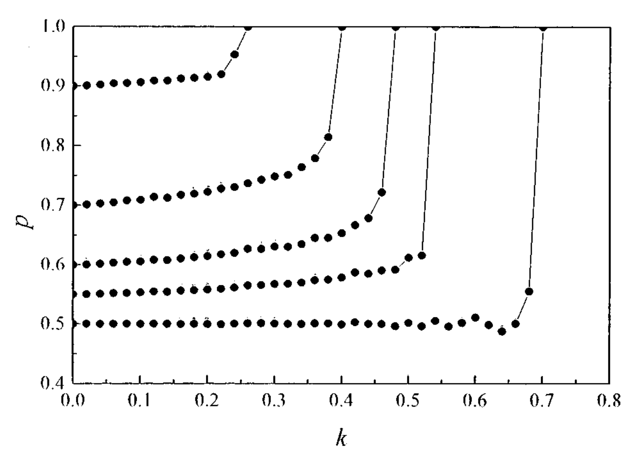

FIG. 6. Asymptotic value of p as a function of k for different values of p_0. Each dot has been obtained from an average over $\sim 10^2$ realizations of the initial condition for a set of 10^3 elements.

the system evolve as a single element. Thus, an important common feature in the evolution of coherent bistable elements and synchronized limit-cycle oscillators is that, as a consequence of global coupling, the details of the individual dynamics are manifested at a macroscopic level.

Since the coupling mechanism considered here is a kind of long-range diffusion process, it is worthwhile to compare the present results with those corresponding to a system of bistable elements coupled through near-neighbor, ordinary diffusion. Such a system is described by the real Ginzburg-Landau equation

$$\partial_t x = D\nabla_{\mathbf{r}}^2 x + x - x^3, \tag{11}$$

where the diffusivity D plays the role of coupling constant. This equation is widely used as a description of the order-parameter dynamics of second-order phase transtions [14]. It is well known that the Ginzburg-Landau equation predicts the formation of spatial domains where the field $x(\mathbf{r},t)$ adopts one of the two stables values $x=\pm 1$. A plane isolated domain wall is in principle stationary but, by interaction between neighboring walls, the domains change their form, move, and eventually, after a long time, coalesce in a homogeneous state, which involves the whole system.

The phase separation observed in the Ginzburg-Landau equation is in principle analogous to the division in two groups of elements observed in the globally coupled system for $k<k_c$. However, two important differences exist. In the first place, global coupling forces the asymptotic state of each element to differ from the stable states of isolated elements, $x_i=\pm 1$. Second, the long-range character of global interactions does not allow domain formation, i.e., spatial ordering, which is instead a typical feature in systems driven by ordinary diffusion. On the other hand, when the coupling constant is greater than the critical value k_c, the whole set of elements form a single domain in a time comparable to the temporal scales of individual evolution.

Besides this connection with the theory of second-order spatially inhomogeneous critical phenomena, as mentioned in the Introduction, the present model of globally coupled bistable elements is related to the description of spin systems and neural networks. This relation has already been discussed in the literature [9,10]. Beyond these applications to physical systems, the model could also be useful in the study of a sociological problem, namely, the problem of public opinion formation and social decision making. Suppose that, at a certain moment in the future, a social group has to opt between two instances by means of individual voting. In the absence of interaction between individuals, it is most likely that a person will maintain his or her present preference for one of the options up to the moment of the election—just as, in the absence of coupling, the variable $x_i(t)$ preserves its sign along the whole evolution. Nowadays, however, individual opinion is strongly exposed to the influence of mass communication media [15]. In the best (fairest) cases, this influence occurs through the publication of opinion surveys and polls during the period previous to the election [16]. No doubt, mass media provide a class of global interaction between individuals and, under its action, a plausible assumption is that the individual opinion is to some extent driven by the average opinion. This assumption is precisely described by the kind of global coupling considered here.

This interpretation of the model of globally coupled bistable elements inspires the proposal of several generalizations that are indeed worth considering. For instance, it would be interesting to analyze the effect of an asymmetry in the potential of Eq. (1), such that only one stationary state is truly stable whereas the other becomes metastable. This intrinsic preference for one of the states can be compared with the evolution in the bistable symmetric potential from an asymmetric initial condition, as described by Eq. (10). The question on the equivalence between these two sources of asymmetry—the potential or the initial condition—arises then quite naturally. A second generalization, which is certainly relevant to the model of opinion formation, is to admit the possibility that the coupling constant is not the same for all the elements, but is chosen at random for each element from a prescribed distribution. In physical models, this form of quenched disorder would represent some kind of spatial inhomogeneity. To the author's knowledge, the effects of inhomogeneities in the coupling strength has not been considered, up to this moment, in the literature on globally coupled systems.

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